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Note

# Vertex critical 4-dichromatic circulant tournaments

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**Abstract**

An infinite family of vertex critical 4-dichromatic circulant tournaments is presented, answering a problem posed by Neumann–Lara and Urrutia (1984).

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Let $D$ be a digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of $D$, respectively. The digraph $D$ is *acyclic* if it contains no directed cycle. A subset of $V(D)$ which induces an acyclic subdigraph of $D$ will be called *acyclic*; $D$ is said to be an *oriented graph* provided it contains no directed cycle of length two.

The *dichromatic number* $\mathrm{dc}(D)$ of a digraph $D$ was defined in [3] (and independently in [2]) as the least number of colours needed to colour the vertices of $D$ in such a way that each chromatic class is acyclic. A digraph $D$ is called *n-dichromatic* if $\mathrm{dc}(D) = n$ and *vertex critical* (v.c.) *n-dichromatic* if $\mathrm{dc}(D) = n$ and $\mathrm{dc}(D - u) < n$ for every $u \in V(D)$.

There is only one v.c. 2-dichromatic tournament: the cyclic triangle, $\vec{C}_3$. In [5] an infinite family of v.c. $r$-dichromatic regular tournaments was constructed for each $r \geqslant 3$, $r \neq 4$ but only one example of a v.c. 4-dichromatic regular tournament was presented. Given three digraphs $D_1, D_2, D_3$, a new digraph $t(D_1, D_2, D_3)$ was defined there, up to isomorphism, as follows: Let $D'_1, D'_2, D'_3$ be three pairwise vertex-disjoint digraphs such that $D'_j \cong D_j$ for $j = 1, 2, 3$.

Then $t(D_1, D_2, D_3)$ is the digraph whose vertex set is $\bigcup_{i=0}^{2} V(D'_i)$ and with arc set $\bigcup_{i=0}^{2} A(D'_i) \cup \{(u,v)\colon\ u \in V(D'_i),\ v \in V(D'_{i+1})\}$, where the indices are taken mod 3.

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The existence of an infinite family of (nonregular) v.c. 4-dichromatic tournaments is an immediate consequence of the fact that $t(\vec{C}_3, \vec{C}_3, T)$ is a (nonregular) v.c. 4-dichromatic tournament provided $T$ is a v.c. 3-dichromatic tournament [5, Theorem 3].

In this paper an infinite family of v.c. 4-dichromatic circulant tournaments is presented, solving a problem of [5].

Let $Z_{2n+1}$ be the set of integers mod $2n+1$ and $J$ a subset of $Z_{2n+1} - \{0\}$ such that for every $w \in Z_{2n+1}$, $w \in J$ if and only if $-w \notin J$. We define the circulant tournament $\vec{C}_{2n+1}(J)$ by $V(\vec{C}_{2n+1}(J)) = Z_{2n+1}$, $A(\vec{C}_{2n+1}(J)) = \{(i,j){:}\ i, j \in Z_{2n+1} \text{ and } j-i \in J\}$.

We recall that the automorphism group of any circulant digraph is vertex transitive.

In [4] it was proved that $\vec{C}_{11}(1, 3, 4, 5, 9)$ is the only 4-dichromatic tournament of minimum order. So it is v. critical. From the fact that $\{\{0,1,4,5\}, \{2,3,6,7\}, \{8,9,10\}\}$ is a partition of $Z_{11}$ into acyclic subsets of $\vec{C}_{11}(1,3,4,5,9)$ — arc 89 and since $\vec{C}_{11}(1,3,4,5,9)$ is arc-transitive (proof of Theorem 2.1 in [4]), it follows that $\vec{C}_{11}(1,3,4,5,9)$ is also arc critical 4-dichromatic. It follows that $\vec{C}_{11}(1,3,4,5,9)$ is the only 4-dichromatic oriented graph of order at most 11.

In [6], Parker and Reid proved that $\vec{C}_{13}(1,2,3,5,6,9)$ which is denoted by $\mathrm{ST}_{13}$, is the only tournament of order 13 (up to isomorphism) which does not contain a transitive tournament on 5 vertices. Therefore, $\mathrm{dc}(\mathrm{ST}_{13}) \geqslant 4$, and since $\{\{0,1,2,3\}, \{4,5,6,7\}, \{8,9,10,11\}, \{12\}\}$ is a partition of the set of vertices into 4 acyclic subsets, it follows that $\mathrm{dc}(\mathrm{ST}_{13}) = 4$ and $\mathrm{dc}(\mathrm{ST}_{13} - \{12\}) = 3$. Then $\mathrm{ST}_{13}$ is a v.c. 4-dichromatic circulant tournament. Let $D_m$ be the circulant tournament defined by

$$D_m = \vec{C}_{6m+1}(1, 2, \ldots, 2m-1, -2m, 2m+1, 2m+2, \ldots, 3m).$$

Clearly, $\{\{0, 1, \ldots, 2m-1\}, \{2m, 2m+1, \ldots, 4m-1\}, \{4m, 4m+1, \ldots, 6m-1\}, \{6m\}\}$ is a partition of $V(D_m)$ into 4 acyclic subsets. Therefore, $\mathrm{dc}(D_m) \leqslant 4$ and $\mathrm{dc}(D_m - \{6m\}) \leqslant 3$. In order to prove that $D_m$ is v.c. 4-dichromatic for $m \geqslant 2$ it is sufficient to prove that $\mathrm{dc}(D_m) \geqslant 4$ for $m \geqslant 2$. Notice that $D_2 = \mathrm{ST}_{13}$. Notice also that $D_1 = \vec{C}_7(1,-2,3) \cong \vec{C}_7(1,2,3)$ is 2-dichromatic.

Let $r$ and $s$ be two integers such that $1 \leqslant s < r$. The tournament $H_{r,s}$ is defined by $V(H_{r,s}) = \{1, 2, \ldots, r+s\}$; $A(H_{r,s}) = \{(i,j){:}\ 1 \leqslant i < j \leqslant r+s \text{ and } j-i \neq r\} \cup \{(i+r, i){:}\ i \leqslant s\}$.

We need the following lemma.

**Lemma.** *The maximum number of vertices in a transitive subtournament of $H_{r,s}$ is $r$.*

**Proof.** Let $S$ be an acyclic set of vertices of $H_{r,s}$. If for every $1 \leqslant i \leqslant s$, $S$ does not contain $\{i, i+r\}$, then $|S| \leqslant r$. If for some $i$, $\{i, i+r\} \subseteq S$, then $k \notin S$ for every $k$ such that $i < k < i+r$, since $\{i, k, i+r\}$ induces a cyclic triangle in $H_{r,s}$. It follows that $|S| \leqslant s+1 \leqslant r$. Since $\{1, 2, \ldots, r\}$ induces a transitive subtournament of $H_{r,s}$ the lemma follows. □

**Theorem.** *$D_m$ is a vertex critical 4-dichromatic circulant tournament for $m \geqslant 2$.*

**Proof.** We only need to prove that $\mathrm{dc}(D_m) \geqslant 4$. To this end, it suffices to prove that every acyclic set of vertices $S$ of $D_m$ has cardinality at most $2m$. Since $D_m$ is vertex transitive, we may assume that 0 is the source of $D_m[S]$. Therefore, $S_1 = S - \{0\} \subseteq N^+(0) = (\{1,2,\ldots,3m\} \cup \{4m+1\}) - \{2m\}$.

Since $D_m[N^+(0)] - \{4m+1\} \cong H_{2m-1,m}$ (the order preserving bijection is an isomorphism), it follows from the lemma that $|S_1| \leqslant 2m-1$ in case $4m+1 \notin S_1$. Assume that $4m+1 \in S_1$.

From the equality

$$A(D_m[N^+(0)]) = A(D_m[N^+(0)] - \{4m+1\}) \cup \{(4m+1, j)\colon\ j = 1,2,\ldots,m,2m+1\}$$
$$\cup \{(m+j, 4m+1)\colon\ j = 1,2,\ldots,m-1,m+2,\ldots,2m\},$$

it follows that either $S_1 \cap \{1,2,\ldots,m\} = \emptyset$ or $S_1 \cap \{m+1,m+2,\ldots,2m-1\} = \emptyset$ for otherwise, $D_m[S_1]$ would contain a cyclic triangle. Similarly, either $2m+1 \notin S_1$ or $S_1 \cap \{2m+2,2m+3,\ldots,3m\} = \emptyset$.

Therefore, if $S_1 \cap \{m+1,m+2,\ldots,2m-1\} \neq \emptyset$ then $|S_1| \leqslant 2m-1$.

Suppose that $S_1 \cap \{m+1,m+2,\ldots,2m-1\} = \emptyset$. If $2m+1 \in S_1$ then $S_1 \subseteq \{1,2,\ldots,m, 2m+1, 4m+1\}$. But since $\{1,2,2m+1\}$ induces a cyclic triangle, then $|S_1| \leqslant m+1 \leqslant 2m-1$. Finally, if $2m+1 \notin S_1$ then $S_1 \subseteq \{1,2,\ldots,m\} \cup \{2m+2,2m+3,\ldots,3m\} \cup \{4m+1\}$. But since $\{j, j+2m+1, 4m+1\}$ induces a cyclic triangle for every $j \in \{1,2,\ldots,m-1\}$, it follows that $|S_1| \leqslant m+1 \leqslant 2m-1$. □

We recall that the v.c. $r$-dichromatic tournaments given in [5] are circulants only for $r = 3$, 5 and 8, so it is natural to propose the following conjecture.

**Conjecture.** There is an infinite family of v.c. $r$-dichromatic circulant tournaments for each $r \geqslant 3$ (only the values $r \geqslant 6$, $r \neq 8$ have to be considered).

## References


[1] P. Erdős, Problems and results in number theory and graph theory, in: Proc. 9th Manitoba Conf. Numer. Math. and Computing (1979) 3–21.

[2] H. Meyniel, Extension du nombre chromatique et du nombre de stabilité, Preprint.

[3] V. Neumann-Lara, The dichromatic number of a digraph, J. Combin. Theory Ser. B 33 (1982) 265–270.

[4] V. Neumann-Lara, The 3 and 4-dichromatic tournaments of minimum order, Discrete Math. 135 (1994) 233–243.

[5] V. Neumann-Lara and J. Urrutia, Vertex critical $r$-dichromatic tournaments, Discrete Math. 40 (1984) 83–87.

[6] K.B. Reid and E.T. Parker, Disproof of a conjecture of Erdős and Moser on tournaments, J. Combin. Theory 9 (1970) 225–238.